LIST OF QUESTIONS

Notice that the questions followed by a '(*)' are more challenging than the others, and will be asked only if the exam is already going well :).

(1) Show that $f : \mathbb{R} \to \mathbb{R}$ is continuous in the topology sense if and only if it is continuous in the analysis sense.
(2) Define the product topology and the metric topology. Show that they agree on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ with the Euclidean metric.
(3) Let $A \subseteq B \subseteq X$, with B open. Show that A is open in B if and only if it is open in X. What if B is not open?
(4) Define connected and path-connected. Show that a path-connected space is connected.
(5) Prove that $\mathbb{Q}$ is totally disconnected.
(6) Show (with pictures) that the Cantor set is totally disconnected.
(7) Is $\mathbb{R}^2 - \mathbb{Q}^2$ connected?
(8) Show that a product of path-connected spaces is path-connected.
(9) Are connected components closed? Prove it. What about path-connected components?
(10) Let $\{C_n\}$ be connected subspaces of X. Suppose that $C_n \cap C_{n+1} \neq \emptyset$ for all n. Show that $\bigcup C_n$ is connected.
(11) Let $f : S^1 \to \mathbb{R}$ be continuous. Show that there exists $(x, y) \in S^1$ so that $f((x, y)) = f((-x, -y))$.
(12) Prove that for any countable set $A \subseteq \mathbb{R}^2$, we have that $\mathbb{R}^2 - A$ is connected.
(13) Prove that $X = \{0, 1\}^{\mathbb{N}}$ is homeomorphic to $X \times X$.
(14) Let X be metric space, and $A, B \subseteq X$ disjoint and closed. Show that there exists a continuous function $f : X \to [0, 1]$ with $f(A) = \{0\}$, $f(B) = \{1\}$.
(15) Prove that the in the line with double zero there is no injective path connecting the two zeros.
(16) Show that the line with double zero is path-connected.
(17) Prove that the cofinite topology on $\mathbb{R}$ is not Hausdorff.
(18) Give an example of a non-Hausdorff space X with a point p so that $X \setminus \{p\}$ is Hausdorff.
(19) Prove that for all pairs of distinct points $\{p, q\} \subseteq S^2$, $S^2 \setminus \{p, q\}$ is homeomorphic to $S^2 \setminus \{(0, 0, 1), (0, 0, -1)\}$.
(20) Let $f : X \to Y$ be continuous, and let Y be Hausdorff. Show that $\{(x, f(x)\} \subseteq X \times Y$ (the graph of the function) is closed.
(21) Give example of continuous bijection which is not a homeomorphism. Can you give a general criterion for showing that a continuous bijection is a homeomorphism?

(22) Define the limit of a sequence. Prove that the limit of a sequence is unique if X is Hausdorff.
(23) Prove that a subspace of a Hausdorff space is Hausdorff. What about normal?
(24) Let X be first-countable. How does one characterise the closure of some $A \subseteq X$ in terms of limits?
(25) Prove that the cofinite topology on $\mathbb{R}$ is not first countable.
(26) State and prove the characterisation of continuity in terms of limits for first-countable spaces.
(27) Let X be first countable, and let (x_n) be a sequence. Also, let x be so that every neighborhood of x contains infinitely many points of the sequence. Show that a subsequence of (x_n) converges to x.
(28) Show that metric spaces are Hausdorff.
(29) Prove that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$.
(30) When is the discrete topology compact?
(31) Prove that closed intervals are compact.
(32) Let X be a topological space and $Y, Z \subseteq X$. Suppose that Y and Z are compact, and that $X = Y \cup Z$. Show that X is compact.
(33) Show that if X is compact and $C \subseteq X$ is closed, then C is compact.
(34) State the characterisation of compactness for metric spaces. Prove one of the implications.
(35) Give at least two proofs that $\mathbb{R}$ is not compact.
(36) Show that a continuous function $f : X \to \mathbb{R}$, where X is compact, has a maximum.
(37) Let X be compact, Y Hausdorff, $f : X \to Y$. Show that f is closed.
(38) Let X be compact and $f : X \to Y$ be continuous and surjective. Prove that Y is compact.
(39) Start the proof that a compact Hausdorff space is normal.
(40) Let X be a compact metric space. Suppose that $f : X \to X$ is continuous and $f(x) \neq x$ for all $x \in X$. Prove that there exists $\epsilon > 0$ so that $d(x, f(x)) > \epsilon$ for all $x \in X$.
(41) Let x_n be a sequence in the topological space X, converging to x. Show that $\{x_n\} \cup \{x\}$ is compact.
(42) Let (X, d) be a compact metric space. Show that $\sup_{x,y\in X} d(x, y)$ is achieved.
(43) Define the quotient topology. Prove it's a topology.
(44) Prove that a quotient of a connected space is connected.
(45) Prove that the Klein bottle is Hausdorff.
(46) Let $p : X \to Y$ be a quotient map, with Y connected and $p^{-1}(\{y\})$ connected for each $y \in Y$. Prove that X is connected.
(47) Define the fundamental group. Prove one of the group axioms.
(48) Give an example of two paths with the same endpoints in some topological space that are not homotopic.
(49) Give an example of two spaces that are homotopy equivalent but not homeomorphic.

(50) What is the fundamental group of the Möbius strip?
(51) To what extent does the fundamental group depend on the basepoint?
(52) Given examples when f_* is neither injective nor surjective.
(53) What is a homotopy equivalence? Give a (sufficiently non-trivial) example.
(54) Outline the proof that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^3$.
(55) Let $p, q \in S^2$ be distinct points. Is $S^2 \setminus \{p\}$ homeomorphic to $S^2 \setminus \{p, q\}$?
(56) Show that if X is simply connected then paths with the same endpoints are homotopic. (You may just draw the homotopy, no formulas needed.)
(57) State the lifting lemma for paths. Describe a map $\pi_1(X, x_0) \to \hat{X}$. When is it surjective/injective?
(58) Outline the proof that the fundamental group of S^1 is $\mathbb{Z}$.
(59) You have two homotopic maps. What can you say about the induced maps at the level of π_1?
(60) Show that $f : S^1 \to S^1$ given by $(x, y) \to (-x, -y)$ is not homotopic to id.
(61) Let $f : D^2 \to D^2$ given by $(x, y) \to (-x, -y)$. Is it homotopic to id?
(62) State van Kampen's Theorem.
(63) What is $\pi_1(D^2/_\sim)$, where $x \sim -x$ for all $x \in S^1$? What about if $x \sim x'$ where x' is rotated by $2\pi/3$ (again $x \in S^1$)?
(64) Using van Kampen, write a presentation of the fundamental group of the 2-torus.
(65) Using van Kampen, write a presentation of the fundamental group of the Klein bottle.
(66) What is the fundamental group of S^3 minus finitely many points?
(67) Let X be a set and let p be an element of X. Check that

$$\tau = \{A \subseteq X \mid p \notin A \text{ or } X - A \text{ is finite}\}$$

defines a topology on X. [sheet 1]
(68) For each $x \in \mathbb{R}$, let $I_x = (x, \infty)$, and let $I_\infty = \emptyset$ and $I_{-\infty} = \mathbb{R}$. Check that

$$\tau = \{I_x \mid x \in \mathbb{R} \cup \{-\infty, \infty\}\}$$

defines a topology on $\mathbb{R}$. [sheet 1]
(69) Define the interior and the closure of a set, and give some characterizations. [sheet 2]
(70) Let $f : X \longrightarrow Y$ and $g : Z \longrightarrow V$ be maps between topological spaces. Define the map

$$f \times g : X \times Z \longrightarrow Y \times V : (x, z) \mapsto (f(x), g(z))$$

Can you state some properties of f and g which are preserved by $f \times g$? (being open maps? being closed maps?) [sheet 2]

(71) Let (X, d) be a metric space equipped with a finite number of points. Show that in X the distance topology coincides with the discrete topology. [sheet 2]
(72) Let Y be a subspace of a topological space X (i.e. Y is a topological space equipped with the subspace topology) and let A be a subset of Y. Let $\text{int}_X(A)$ be the interior of A with respect to X and $\text{int}_Y(A)$ be the interior of A with respect to Y. Show that $\text{int}_X(A) \subseteq \text{int}_Y(A)$ and give an example of when the equality does not hold. [sheet 2]
(73) Let X be a topological space equipped with a topology T_X. Let be Y a subset of X, and let T_Y be the subset topology on Y with respect to T_X. Let Z be a subset of Y, let $T_{Z,Y}$ be the subset topology on Z with respect to T_Y and let $T_{Z,X}$ be the subset topology on Z with respect to T_X. Show that $T_{Z,Y} = T_{Z,X}$. [sheet 2]
(74) Is the product of two closed sets closed? [sheet 3]
(75) Give some examples of homeomorphic and not homeomorphic subsets of $\mathbb{R}^n$ [sheet 3]
(76) Can you give an example of a disconnected set whose closure is connected?
(77) What are the connected subsets of a space X endowed with discrete topology? [sheet 3]
(78) Show that the product of path-connected spaces is path-connected. [sheet 3]
(79) What are the connected subsets in $\mathbb{R}$ with standard topology? [sheet 4]. From the fact that [a,b] with $a \leq b$ is connected, deduce the intermediate value theorem. [sheet 3]
(80) Give the definition, an example and a characterization of totally disconnectedness. [sheet 4]
(81) State a characterization of compactness in terms of the finite intersection property [sheet 4]
(82) Describe the construction of the Cantor set with a picture, and state some properties of it.
(83) Give some examples of compact and non-compact spaces. Can you provide a compact topology on $\mathbb{R}$? [sheet 5]
(84) Is the product of Hausdorff spaces still Hausdorff? [sheet 5]
(85) Define the Hausdoff property and give some examples. Show that subspace of Hausdorff space is Hausdorff. [sheet 4,7]
(86) Describe the line with two zeros, and state some properties.
(87) Define first and second countable spaces. Can you provide some examples? [sheet 5]
(88) Let Z be a complete metric space and let Y be a subset of Z. Show that Y is complete if and only if it is closed. [sheet 6]
(89) State some properties which are topological invariants.
(90) Describe a homeomorphism between the n-dimensional sphere without a point $S^n \setminus \{p\}$ and $\mathbb{R}^n$ [sheet 7]

(91) Let $A \subseteq X$ be a discrete subset of a compact space. Is A finite? If not, which additional condition makes it true? [sheet 8]
(92) Which properties are preserved by quotient topology? which not? Provide some examples. [sheet 8]
(93) Let X be a topological space and $q : X \longrightarrow Y$ a quotient map. Let $f : Y \longrightarrow Z$ be any function. Prove that f is continuous if and only if $f \circ q$ is continuous. [sheet 9]
(94) Describe a strategy to show that if X is a Hausdorff space and $K \subseteq X$ is compact, then the quotient X/K is Hausdorff. [sheet 9]
(95) Let X be a topological space, and let Δ be the diagonal of $X \times X$. Show that X is Hausdorff if and only if Δ is closed in $X \times X$.
(96) Let X be Hausdorff and let $\sim$ be an equivalence relation on X. Let R be the graph of the relation, i.e.

$$R = \{(x, y) \in X \times X : x \sim y\}$$

Suppose moreover that the quotient map $p : X \longrightarrow X/\sim$ is open. Show that if $X/\sim$ is Hausdorff then R is closed in $X \times X$. [sheet 9]
(97) Let $f : X \longrightarrow Y$ be a map, and let Y be compact and Hausdorff. Show that f is continuous if and only if the graph of f is closed in $X \times Y$.
(98) Can you write the interval $[a, b]$ as a quotient of (c, d)? Viceversa? [sheet 9]
(99) Show that a convex subset of $\mathbb{R}^n$ is contractible.
(100) Let X and Y be topological spaces and let $x \in X, y \in Y$. Consider the map

$$\begin{aligned} f : \pi_1(X \times Y, (x, y)) &\longrightarrow \pi_1(X, x) \times \pi_1(Y, y) : \\ : [\gamma] &\mapsto ([p_X \circ \gamma], [p_Y \circ \gamma]) \end{aligned}$$

where p_X and p_Y are the projections of $X \times Y$ in X and Y, respectively. Show that f is indeed a well defined map. [sheet 10]
(101) Show that f as in 100 is a homomorphism. [sheet 10]
(102) Sketch the proof that f as in 100 is bijective. [sheet 10]
(103) Let X be a path connected space. What can you say about the fundamental group of X? Write down explicitly the isomorphism $\pi_1(X, x) \cong \pi_1(X, y)$ for $x, y \in X$.
(104) Show that a contractible space is path-connected. [sheet 11]
(105) Give two examples of covering of the circle. [sheet 11]
(106) Let $p : X \longrightarrow Y$ and $q : Y \longrightarrow Z$ be two covering maps. Assume moreover that all the fibers of q are finite. Describe a strategy to prove that $q \circ p$ is a covering map. [sheet 11]
(107) Let $f : X \longrightarrow Y$ be a map. Show that the induced map f_* between the respective fundamental groups is (well defined and) an homomorphism. [sheet 12]

(108) Let X be path connected. Show that X is contractible if and only if for any path connected top space Y and any pair of functions f, g from X to Y, we have that f and g are homotopic. [sheet 12]
(109) Give an example of connected but not path-connected topological space
(110) Give an example of a path-connected space such that $\pi_1(X) = \{e\}$ but is not contractible.
(111) Describe a path-connected finite topological space not with the trivial topology. (*)
(112) Describe a topological space with fundamental group $\mathbb{Z}/5$. (*)
(113) Let $\pi : X \times Y \to X$ be the natural map, and suppose that Y is compact. Show that π is closed. (Hint: $x \in X - \pi(Z)$ means that $(x, y) \notin Z$ for all $y \in Y$.) (*)
(114) Let X be normal, and $A, B \subseteq X$ disjoint and closed. Try to describe a strategy to prove that there exists a continuous function $f : X \to [0, 1]$ with $f(A) = \{0\}$, $f(B) = \{1\}$. (*)
(115) Describe a strategy to prove that the cylinder is not homeomorphic to the Möbius strip. (*)
(116) Let X be path-connected. There is a bijective correspondence between conjugacy classes in the fundamental group and homotopy classes of maps $S^1 \to X$. Can you guess what it is? (*)